Serie n°3 – September 24th

Complex Numbers
Reciprocal spaces, X-ray diffraction

Exercise 1 :

1a. What are the real and imaginary parts of: *(i)* $\sqrt{-16}$; *(ii)* $\frac{1+3i}{2+i}$

1b. Find the modulus and argument of the following complex numbers, and express in the polar form: *(i)* $\frac{1}{\sqrt{2}}(1+i)$; *(ii)* $\frac{1+i\sqrt{3}}{1+i}$

1c. Simplify the following complex expressions: $\left(\frac{1+i}{1-i}\right)^{2024}$

1d.

(i) Using Euler's relation, show that:

$$\forall (x,y) \in \mathbb{R}^2, sin(x+iy) = sin(x)\, ch(y) + icos(x)sh\,(y)$$

(Reminder: $\forall y \in \mathbb{R}, ch(y) = \frac{e^y+e^{-y}}{2}$ and $sh(y) = \frac{e^y-e^{-y}}{2}$)

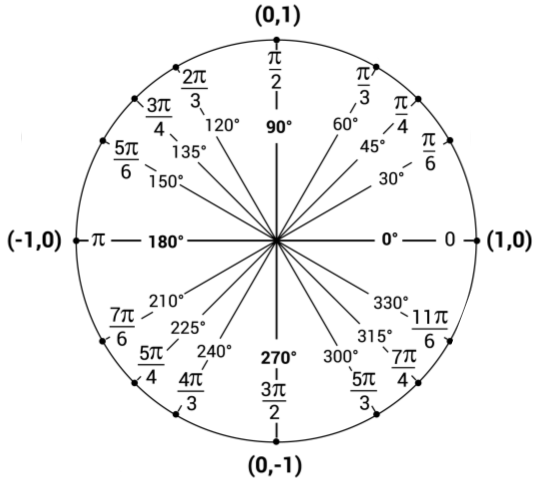
(ii) Use this expression to find one solution of the equation in $\mathbb{C}$: $sin(z) = 2$.

Exercise 2 : Trigonometry and unit circle

2a. Place with a cross the following complex numbers on the unit circle below:

(i) $e^{i\frac{7\pi}{4}}$; (ii) $\frac{1}{\sqrt{2}}(1-i)$; (iii) $z = i \times \frac{\sqrt{3}+i}{2}$;

2b. Calculate the roots of the equation: $z^3 = \cos\left(\frac{3\pi}{2}\right) + isin(\frac{3\pi}{2})$. Place them on the unit circle and show that they form an equilateral triangle.



Exercise 3: Complex index of refraction and absorption coefficient

We consider a dielectric material inside which a monochromatic plane wave with a frequency in the Infrared is traveling. The frequency turns out to correspond to a vibrational mode of the crystal lattice, leading to a transfer of energy between the electromagnetic wave and the medium. The wave is hence losing energy as it is traveling, which we can approach by considering a complex dielectric constant and index of refraction. We then define:

$$\varepsilon_r = \varepsilon_1 + i\varepsilon_2 \text{ and } n = n_1 + in_2$$

Where ε_1 and ε_2 are the real and imaginary parts of the dielectric constant respectively and n_1 and n_2 are the real and imaginary parts of the index of refraction, respectively. We remind you that $n = \sqrt{\varepsilon_r}$

3a. Show that we must have: $\begin{cases} n_1^2 - n_2^2 = \varepsilon_1 \\ 2n_1n_2 = \varepsilon_2 \end{cases}$

3b. Deduce that: $\begin{cases} n_1 = \frac{1}{\sqrt{2}}\sqrt{\left(\varepsilon_1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)} \\ n_2 = \frac{1}{\sqrt{2}}\sqrt{\left(-\varepsilon_1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)} \end{cases}$

3c. We consider a plane wave of angular frequency ω and vector $k = \frac{2\pi}{\lambda}n$, λ being the wavelength, traveling along the x axis: $\vec{E}(x,t) = \overrightarrow{E_0}e^{i(kx-\omega t)}$.

Knowing that the intensity at a point x in the material is $I(x) = \left|\overrightarrow{E(x,t)}\right|^2$, show that :

$$I = I(0)e^{-\alpha x} \text{ with } \alpha = \frac{4\pi}{\lambda}n_2$$

3d. Absorption and attenuation

(i) Gallium Arsenide (GaAs) is a material widely used in optoelectronics and particularly in photovoltaics, in part because of its high absorption in the visible. At $600\ nm = 6 \times 10^{-7}\ m$, $n_2 = 0.6$.

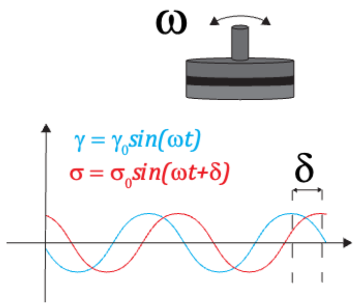
What is the penetration depth δ_{GaAs} at 600 nm, that is the length of propagation inside the materials for which $\frac{I(\delta)}{I(0)} = \frac{1}{e}$?

(ii) Light in an silica optical fiber propagates in the core with a very low attenuation of the glass at the telecommunication wavelength $\lambda = 1.55\ \mu m$. Typically, amplifiers are put every 50 km of fiber to reinforce the signal. This corresponds to a loss of 90% of intensity of the signal: $\frac{I(50\ km)}{I(0)} = 0.1$

Calculate what is the imaginary index of refraction of silica at $\lambda = 1.55\ \mu m$?

Exercise 4 : Rheology and complex numbers

Dynamic or oscillatory rheology is a powerful technique to characterize the viscoelastic properties of materials (e.g. its viscosity η). A schematic is illustrated on the right picture: a cylindrical sample is placed between two parallel plates and a sinusoidal deformation (stress or strain) is applied to the material through the oscillation of the top plate at a fixed frequency ω. The material response (strain or stress) is then measured.



$G' = (\sigma_0/\gamma_0)cos(\delta) \leftrightarrow$ Elasticity
$G'' = (\sigma_0/\gamma_0)sin(\delta) \leftrightarrow$ Viscous Nature

Let's assume that a sinusoidal strain γ^* is applied and that the stress τ^* is measured, both of them can be expressed using the complex exponential function: $\gamma^* = \gamma_0 e^{i\omega t}$, $\tau^* = \tau_0 e^{i(\omega t+\delta)}$.

With δ the phase shift between the deformation and the response ($\delta = 0$ for a purely elastic material, $\delta = \frac{\pi}{2}$ for a purely viscous one, and $0 < \delta < \frac{\pi}{2}$ for viscoelastic materials).

4a. In the linear regime, recall that the following relation links γ^* and τ^*: $\tau^* = G^*\gamma^*$ where $G^* = G' + iG''$ is the complex modulus. Show that:

$$G' = \frac{\tau_0}{\gamma_0}\cos(\delta) \text{ and } G'' = \frac{\tau_0}{\gamma_0}\sin(\delta)$$

4b. G' and $G''$are respectively the storage and loss modulus.

(i) In order to measure the material damping properties, the value of $\tan(\delta)$ can be computed. Show that $\tan(\delta) = \frac{G''}{G'}$.

(ii) What value of $\tan(\delta)$ would you expect for a purely elastic material ? and a purely viscous material ? If you consider now a viscoelastic material, what can you conclude on the meaning of G' and G''?

4c. Express the value of the complex viscosity extracted from oscillatory rheology as a function of G' and G'' and conclude that it can be written as $\eta^* = \eta' + i\eta''$. (Recall: $\tau = \eta \times \dot{\gamma} = \eta \times \frac{d\gamma}{dt}$).

4d. Show that $|\eta^*| = \frac{|G^*|}{\omega}$

4e. Now assume that you are applying a sinusoidal stress $\tau^* = \tau_0 e^{i\omega t}$ and you measure the response strain $\gamma^* = \gamma_0 e^{i(\omega t+\delta)}$. Compute once again the complex modulus G^* and the complex viscosity η^* and show that we still have $|\eta^*| = \frac{|G^*|}{\omega}$.